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Problem 43

Problem. Set up and evaluate the definite integral for the area of the surface generated by revolving the curve $y = \sqrt[3]{x} + 2$ about the y-axis.

Solution. Note that the curve is revolved about the y-axis, not the x-axis. It will be easier if we reverse the roles of x and y. Then the function is $x = (y-2)^3$ and

$$x' = 3(y-2)^2.$$

Then

$$\sqrt{1 + (x')^2} = \sqrt{1 + 9(y-2)^4}.$$

According to the drawing, x goes from 1 to 8, so y goes from 3 to 4. The surface area is

$$S = \int_3^4 2\pi(y-2)^3\sqrt{1+9(y-2)^4}\,dy.$$

Let $u = y - 2$ and $du = dy$. Then

$$S = 2\pi \int_1^2 u^3\sqrt{1+9u^4}\,du.$$

Now let $v = 1 + 9u^4$ and $dv = 36u^3\,du$. Then

$$\begin{aligned} S &= \frac{2\pi}{36}\int_1^2 36u^3\sqrt{1+9u^4}\,du \\ &= \frac{\pi}{18}\int_{10}^{145}\sqrt{v}\,dv \\ &= \frac{\pi}{18}\left[\frac{2}{3}v^{3/2}\right]_{10}^{145} \\ &= \frac{\pi}{27}\left(145^{3/2} - 10^{3/2}\right) \\ &= \frac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right). \end{aligned}$$

Problem 44

Problem. Set up and evaluate the definite integral for the area of the surface generated by revolving the curve $y = 9 - x^2$ about the y-axis.

Solution. The curve is revolved about the y-axis, so we will (again) reverse the roles of x and y. (The book used a somewhat different approach in Example 7.) The function is $x = \sqrt{9-y}$.

$$\begin{aligned} x' &= -\frac{1}{2}(9-y)^{-1/2} \\ &= -\frac{1}{2\sqrt{9-y}}. \end{aligned}$$

Then

$$\sqrt{1+(x')^2} = \sqrt{1 + \frac{1}{4(9-y)}}.$$

The surface area is

$$\begin{aligned} S &= \int_0^9 2\pi\sqrt{9-y}\sqrt{1+\frac{1}{4(9-y)}}\,dy \\ &= 2\pi\int_0^9 \sqrt{(9-y)+\frac{1}{4}}\,dy. \end{aligned}$$

At this point, it might be helpful to use the substitution $u = 9 - y$ and $du = -du$. We get

$$\begin{aligned} S &= -2\pi\int_0^9 \sqrt{(9-y)+\frac{1}{4}}\,(-dy) \\ &= -2\pi\int_9^0 \sqrt{u+\frac{1}{4}}\,du \\ &= 2\pi\int_0^9 \sqrt{u+\frac{1}{4}}\,du \end{aligned}$$

One more substitution: let $v = u + \frac{1}{4}$ and $dv = du$. Then

$$\begin{aligned}
S &= 2\pi \int_{1/4}^{37/4} \sqrt{v}\, dv \\
&= 2\pi \left[\frac{2}{3} v^{3/2} \right]_{1/4}^{37/4} \\
&= \frac{4\pi}{3} \left(\left(\frac{37}{4} \right)^{3/2} - \left(\frac{1}{4} \right)^{3/2} \right) \\
&= \frac{4\pi}{3} \left(\frac{37^{3/2}}{8} - \frac{1}{8} \right) \\
&= \frac{4\pi}{3} \left(\frac{37\sqrt{37} - 1}{8} \right) \\
&= \frac{(37\sqrt{37} - 1)\pi}{6}.
\end{aligned}$$

Problem 47

Problem. Use the integration capabilities of a graphing utility to approximate the surface are of the solid of revolution of $y = \sin x$ about the x-axis over $[0, \pi]$.

Solution. We have $y' = \cos x$, so the integral representing the surface area is

$$\int_0^\pi 2\pi \sin x \sqrt{1 + \cos^2 x}\, dx.$$

The TI-83 reports this value to be 14.4236.

Problem 54

Problem. (a) Given a circular sector with radius L and central angle θ, show that the area of the sector is given by

$$S = \frac{1}{2} L^2 \theta.$$

(b) By joining the straight-line edges of the sector in part (a) a right circular cone is formed and the lateral surface area of the cone is the same as the area of the sector. Show that the area is $S = \pi r L$, where r is the radius of the base of the cone.

(c) Use the result of part (b) to verify that the formula for the lateral surface area of the frustum of a cone with slant height L and radii r_1 and r_2 is $S = \pi(r_1 + r_2)L$.

Solution. (a) The area of the full circle is πL^2. With the sector taken out, the remaining area is a fraction $\frac{\theta}{2\pi}$ of the full circle. Thus, the area is

$$\begin{aligned} S &= (\pi L^2) \cdot \frac{\theta}{2\pi} \\ &= \frac{1}{2} L^2 \theta. \end{aligned}$$

(b) If the base radius of the cone is r, then the circumference of the base is $2\pi r$. However, when the cone is slit and flattened, then it is a portion of a circle of radius L, whose circumference is $2\pi L$. So it represents a fraction $\frac{2\pi r}{2\pi L} = \frac{r}{L}$ of the full circle. Thus, the angle, in radians, of the (shaded) region is $\theta = 2\pi \cdot \frac{r}{L}$. So the surface area of the cone is

$$\begin{aligned} \frac{1}{2} L^2 \theta &= \frac{1}{2} L^2 \left(\frac{2\pi r}{L} \right) \\ &= \pi r L. \end{aligned}$$

(c) The surface area of the frustum is the surface area of a full cone of base radius r_2 and lateral height $L + h$ (where h is the remaining distance to the vertex) minus the surface area of a cone of base radius r_1 and lateral height h. So the surface area is

$$\begin{aligned} S &= \pi r_2 (L + h) - \pi r_1 h \\ &= \pi \left(r_2 L + r_2 h - r_1 h \right) \\ &= \pi \left(r_2 L + (r_2 - r_1) h \right). \end{aligned}$$

However, by similar triangles (cross section),

$$\begin{aligned} \frac{r_2}{L + h} &= \frac{r_1}{h}, \\ r_2 h &= r_1 (L + h) \\ &= r_1 L + r_1 h, \\ (r_2 - r_1) h &= r_1 L. \end{aligned}$$

Therefore,

$$\begin{aligned} S &= \pi\left(r_2 L + r_1 L\right) \\ &= \pi(r_1 + r_2)L. \end{aligned}$$

Problem 56

Problem. A right circular cone is generated by revolving the region bounded by $y = hx/r$, $y = h$, and $x = 0$ about the y-axis. Verify that the lateral surface area of the cone is $S = \pi r\sqrt{r^2 + h^2}$.

Solution. The quick way to do this is to use the Pythagorean Theorem to note that $L = \sqrt{r^2 + h^2}$ and then use part (b) of problem 54. Done.

On the other hand, using integration, $y' = \frac{h}{r}$, so

$$\begin{aligned} S &= \int_0^r 2\pi x\sqrt{1 + \frac{h^2}{r^2}}\, dx \\ &= 2\pi\sqrt{1 + \frac{h^2}{r^2}} \int_0^r x\, dx \\ &= 2\pi\sqrt{1 + \frac{h^2}{r^2}} \left[\frac{1}{2}x^2\right]_0^r \\ &= 2\pi\sqrt{1 + \frac{h^2}{r^2}} \cdot \frac{1}{2}r^2 \\ &= \pi r\sqrt{r^2 + h^2}. \end{aligned}$$

Problem 58

Problem. Find the area of the zone of a sphere formed by revolving the graph of $y = \sqrt{r^2 - x^2}$, $0 \le x \le a$, about the y-axis.

Solution. We have

$$y' = -\frac{x}{\sqrt{r^2 - x^2}}.$$

So the surface area is

$$\begin{aligned} S &= \int_0^a 2\pi x\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx \\ &= 2\pi\int_0^a x\sqrt{\frac{r^2}{r^2-x^2}}\,dx \\ &= 2\pi\int_0^a \frac{xr}{\sqrt{r^2-x^2}}\,dx \\ &= 2\pi r\int_0^a \frac{x}{\sqrt{r^2-x^2}}\,dx \end{aligned}$$

Let $u = r^2 - x^2$ and $du = -2x\,dx$. Then

$$\begin{aligned} S &= -\pi r\int_0^a \frac{-2x}{\sqrt{r^2-x^2}}\,dx \\ &= -\pi r\int_{r^2}^{r^2-a^2} \frac{1}{\sqrt{u}}\,du \\ &= \pi r\int_{r^2-a^2}^{r^2} u^{-1/2}\,du \\ &= \pi r\left[2u^{1/2}\right]_{r^2-a^2}^{r^2} \\ &= \pi r\left(2r-2\sqrt{r^2-a^2}\right) \\ &= \pi r\left(2r-2\sqrt{r^2-a^2}\right) \\ &= 2\pi r\left(r-\sqrt{r^2-a^2}\right). \end{aligned}$$

Note that if $a = r$, then the surface area is $S = 2\pi r^2$ which is the surface of half a sphere of radius r.

Problem 65

Problem. Find the area of the surface formed by revolving the protion oin the first quadrant of the graph of $x^{2/3} + y^{2/3} = 4$, $0 \le x \le 8$, about the y-axis.

Solution. As in an earlier problem, we find

$$\begin{aligned} y' &= (4-x^{2/3})^{1/2}x^{-1/3} \\ &= \frac{\sqrt{4-x^{2/3}}}{x^{1/3}}. \end{aligned}$$

and

$$\begin{aligned}\sqrt{1+(y')^2} &= \sqrt{1+\frac{4-x^2/3}{x^{2/3}}} \\ &= \sqrt{1+\frac{4}{x^{2/3}}-1} \\ &= \sqrt{\frac{4}{x^{2/3}}} \\ &= \frac{2}{x^{1/3}}.\end{aligned}$$

Then the surface area is

$$\begin{aligned}S &= \int_0^8 2\pi x \cdot \frac{2}{x^{1/3}}\ dx \\ &= 4\pi \int_0^8 x^{2/3}\ dx \\ &= 4\pi \left[\frac{3}{5}x^{5/3}\right]_0^8 \\ &= \frac{12\pi}{5}\cdot 32 \\ &= \frac{384\pi}{5}.\end{aligned}$$